THE MAGIC SQUARES
of
BENJAMIN FRANKLIN

THE FIRST OF A SERIES OF FOUR PAPERS DESCRIBING THE TECHNIQUE OF LEONHARD EULER APPLIED TO THE LAHIREIAN METHOD OF FORMING MAGIC SQUARES OF ALL SIZES UNDER THE GENERAL TITLE

THE INTRINSIC HARMONY OF NUMBER

by

CLARENCE C. MARDER

ALSO THE SIMPLE METHOD OF RAISING ANY 8 x 8 SQUARE TO THE 16 x 16 SIZE WITH THE AID OF AN AUXILIARY SQUARE PRESERVING ALL THE QUALITIES OF THE ORIGINAL

Three entirely new methods of producing
THE BENT DIAGONALS OF BENJAMIN FRANKLIN

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MAGIC SQUARES AND OTHER PROBLEMS ON A CHESSBOARD

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"The construction of magic squares is an amusement of great antiquity; we hear of their being constructed in India and China before the Christian era, while they appear to have been introduced into Europe by Moscopulus who flourished at Constantinople early in the fifteenth century.

"However, what was at first merely a practice of magicians and talisman makers has now for a long time become a serious study for mathematicians. Not that they have imagined that it would lead them to anything of solid advantage, but because the theory was seen to be fraught with difficulty, and it was considered possible that some new properties of numbers might be discovered which mathematicians could turn to account. This has in fact proved to be the case, for from a certain point of view the subject has been found to be algebraical rather than arithmetical and to be intimately connected with great departments of science such as the 'infinitesimal calculus,' the 'calculus of operations' and the 'theory of groups.'

"No person living knows in how many ways it is possible to form a magic square of any order exceeding 4 x 4. The fact is that before we can attempt to enumerate magic squares we must see our way to solve problems of a more simple character.

"To say and to establish that problems of the general nature of the magic square are intimately connected with the infinitesimal calculus and the calculus of finite differences is to sum the matter up."

An excerpt from a paper published in Proceedings of the Royal Institution of Great Britain. Vol. XVII, No. 96, pp. 50-61, Feb. 4, 1892.

[illegible]

A magic square consists of a series of numbers so arranged in a square, that the sum of each row and column and of both corner diagonals shall be the same amount, which may be termed the summation and is usually indicated by S. Those of the eighth order - the 8 x 8 size - may have other "properties" as Franklin termed them, especially his "bent-diagonals."

A famous French mathematician, Gabriel Phillippe de La Hire (1677-1719) originated the method of using two "primary squares" to form the magic square. A "root square" is substituted for one of them and then the two are added together, forming the magic square, comprising the numbers 1 to 64 in squares of the eighth order.

A contemporary of Benjamin Franklin, the Swiss mathematician Leonhard Euler (1706-1790) used the LaHireian method but (in effect) made the use of the root square unnecessary by placing both primary squares in one cell block, thus forming a magic square with 11 for unity instead of one.

The "magic" is, naturally, in the primary square. The harmonious distribution of eight sets of the digits 1 2 3 4 5 6 7 8 in an 8 x 8 square creates the instruments with which the various forms are constructed. The position of the complementary pairs 1-8 2-7 3-6 4-5 determine the form of "association." Usually the primary is turned one-quarter around to form the second one - primary B - and then the two entered in the same cell block form the Euler-LaHireian square and the complementary pairs are then 11-88 12-87 13-86 and so on to the center pairs 48-51.

Six of these forms of "association" are shown in the Euler-LaHireian squares (829) to (834) and two others - the knight (only) in (814) and the associated (815).

We will proceed with a description of the various forms of the magic rectangle, this being the method used by La Hire to build his primary square, and then on to two other methods of forming the primary square, which, as far as we know, have never appeared in print.

THE FRANKLIN-KNIGHT-NASIK MAGIC RECTANGLES

Gabriel Phillippe de La Hire (1677-1719) was in his sixteenth year when the following came from the pen of La Loubere:- "In these Indian squares it is necessary not merely that the summation of the rows, columns and diagonals should be alike, but that the sum of any eight numbers in one direction as in the moves of a bishop or a knight should also be alike." This "continuous" or "pandiagonal" quality in an 8 x 8 square is generally referred to as Knight-Nasik.[1]

Early in the eighteenth century de La Hire found he could construct these "Indian" squares with one primary square formed with four identical 2 x 8 magic rectangles, the numbers being used in their natural order - 1234 - followed by their complements - 8765 - the second line containing the complements of the first line. The primary square was formed with four of the rectangle (800) and turned one-quarter to the right or left to form the second one.

Half a century later, Franklin, using the more difficult "direct method" formed an 8 x 8 square with his now famous "bent-diagonals."

Had he used the method of La Hire and simply transposed the digits 3 and 4 he could have formed a magic square with the "Indian" quality and his bent-diagonals as well.

There are only three groups of numbers which can be used for this purpose - 1-2-4-3 * 1-2-6-5 * 1-3-7-5 and within each group the "pairing" is alternate as indicated in H (805). Thus they form twelve diverse 2 x 4 magic rectangles (835) which can be reflected, inverted and this inversion reflected, making 48 in all.

The complete table of 2 x 8 magic rectangles should be studied thoroughly.

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1. NASIK. The town of Nasik near Bombay, India, was the home of the Rev. A. H. Frost who contributed a mass of material relating to magic squares and cubes to the South Kensington Museum in London, and his name for the "pandiagonal" or "continuous" quality in any magic square has been generally accepted.

THE TABLE OF 2 x 8 MAGIC RECTANGLES

- - - - - - - - - - -

Here are shown the twelve entirely diverse 2 x 8 Franklin-Knight-Nasik magic rectangles. Each one has been formed with a 2 x 4 rectangle, inverted to form the second half. Each one can be reflected, inverted, turned one-half around, (the reflection inverted) or 48 in all.

Each one can be changed into a Franklin-JAINA rectangle by transposing the second and third 2 x 2 sections. The three numbers 11 - 31 - 51 have been so treated and are shown as 011 - 015 in group 1584, 031 - 035 in group 1386 and 051-055 in group 1287. The other nine - numbers 15 to 65 - can be transposed in the same manner.

Likewise the Knight-Nasik groups numbers 111 to 165 can be transformed into the Franklin-JAINA forms 021-025 041-045 061-065, also being kept in the three groups numbers 1584 - 1386 - 1287 as otherwise there would be a duplication of digits. Thus there are three distinct groups of the Franklin-JAINA forms, each containing eight 2 x 8 rectangles, either 2 x 4 half of which can be reflected, etc., without reference to the other half. The primary squares formed from these groups can be used together.

Note the construction of this Franklin-JAINA form. Each 2 x 4 is made with a 2 x 2 inverted to form the second half. The construction of the thirteenth century JAINA square is shown in (806).[1]

The Knight-Nasik rectangles numbers 111 to 165 can be made into the "associated" form by reflecting the left 2 x 4 section (instead of inverting it). Three examples are shown - 71 - 81 - 91. This transformation is shown in the two Euler-[illegible] magic squares (81[illegible]) - (815).

1. JAINA. In 1904, Professor Schilling contributed to the Mathematical Society of Göttingen the fact that Professor Kielhorn had found a JAINA inscription of the twelfth or thirteenth century in the city of KHAJURAHO, India. This consisted of a 4 x 4 magic square with the particular qualities shown in (806).

FRANKLIN-KNIGHT-NASIK

11	1 5 6 2 8 4 3 7	8 4 3 7 1 5 6 2
31	1 3 4 2 8 6 5 7	8 6 5 7 1 3 4 2
51	1 2 4 3 8 7 5 6	8 7 5 6 1 2 4 3
15	1 5 7 3 8 4 2 6	8 4 2 6 1 5 7 3
35	1 3 7 5 8 6 2 4	8 6 2 4 1 3 7 5
55	1 2 6 5 8 7 3 4	8 7 3 4 1 2 6 5
21	5 1 2 6 4 8 7 3	4 8 7 3 5 1 2 6
41	3 1 2 4 6 8 7 5	6 8 7 5 3 1 2 4
61	2 1 3 4 7 8 6 5	7 8 6 5 2 1 3 4
25	5 1 3 7 4 8 6 2	4 8 6 2 5 1 3 7
45	3 1 5 7 6 8 4 2	6 8 4 2 3 1 5 7
65	2 1 5 6 7 8 4 3	7 8 4 3 2 1 5 6

ALLOWANCE IS MADE FOR NUMBERS OF OTHER 3 FORMS

FRANKLIN-JAINA

011	1 5 8 4	8 4 1 5
015	6 2 3 7	3 7 6 2
031	1 3 8 6	8 6 1 3
035	4 2 5 7	5 7 4 2
051	1 2 8 7	8 7 1 2
055	4 3 5 6	5 6 4 3

GROUP 1584

GROUP 1386

GROUP 1287

ADDITIONAL FRANKLIN-JAINA

021	1 4 8 5	8 5 1 4
025	6 7 3 2	3 2 6 7
041	1 6 8 3	8 3 1 6
045	4 7 5 2	5 2 4 7
061	1 7 8 2	8 2 1 7
065	4 6 5 3	5 3 4 6

KNIGHT-NASIK

111	1 4 6 7 8 5 3 2	8 5 3 2 1 4 6 7
131	1 6 4 7 8 3 5 2	8 3 5 2 1 6 4 7
151	1 7 4 6 8 2 5 3	8 2 5 3 1 7 4 6
115	1 4 7 6 8 5 2 3	8 5 2 3 1 4 7 6
135	1 6 7 4 8 3 2 5	8 3 2 5 1 6 7 4
155	1 7 6 4 8 2 3 5	8 2 3 5 1 7 6 4
121	4 1 6 7 5 8 3 2	5 8 3 2 4 1 6 7
141	6 1 4 7 3 8 5 2	3 8 5 2 6 1 4 7
161	7 1 4 6 2 8 5 3	2 8 5 3 7 1 4 6
125	4 1 7 6 5 8 2 3	5 8 2 3 4 1 7 6
145	6 1 7 4 3 8 2 5	3 8 2 5 6 1 7 4
165	7 1 6 4 2 8 3 5	2 8 3 5 7 1 6 4

ASSOCIATED

71	1 4 6 7 7 6 4 1	8 5 3 2 2 3 5 8
81	1 6 4 7 7 4 6 1	8 3 5 2 2 5 3 8
91	1 7 4 6 6 4 7 1	8 2 5 3 3 5 2 8

QUA[illegible] OF THE MAG[illegible] RECT[illegible]

[illegible] ic re[illegible] in which the right [illegible] "kn[illegible]ght" [illegible].

[illegible]
Jaina [illegible]
mary square[illegible] in [illegible]
the order [illegible]; the [illegible] pair [illegible]

[illegible] st [illegible] 8 magic rectang[illegible] is shown in ([illegible]). [illegible] foll[illegible] in the two lo[illegible]r primary squares in (807) [illegible] of the primary

It [illegible] (B[illegible] the correct totals [illegible] 1243 - 1265 - 1375. See ([illegible]) where the Fra[illegible] covers the [illegible] mak-i[illegible]g the total for the eight numbers 40 instead of 36 the correct total in ([illegible]).

All of these forms can be used together - one for primary A and the other - turned right or left (or reflected and turned) for the second or primary B. For example: the combination of the primary squares made with the rectangles (802) and (803) forms a magic square, but not having these qualities in common both the knight and Franklin qualities are lost. See the example (808).

In the combination of the primary squares made with the rectangles (801) and (802) one quality - the bent diagonal - is retained but the knight quality is lost. Note the bent diagonal in the example (809).

The construction of the thirteenth century JAINA square is shown in (806). The squares made with the Franklin-Jaina rectangles retain the complementary pairs [illegible] each [illegible] subsquare. Refer to the magic square (811) and note that the pairs a[illegible]d [illegible] they are in diagonally opposite [illegible] This [illegible] is [illegible] 8 x 8 squares [illegible] knight [illegible]

In both of these sq[illegible] ([illegible]) and ([illegible]) the pairing is a composite of the Jaina and knight pairing. The complementary pairs will be found in the diagonally opposite corners of eight 3 x 5 rectangles in each horizontal or vertical half, depending on which primary is used for the primary A.

A magic square is called "associated" when the two numbers making up the complementary pairs are diametrically equidistant from the center. In the squares of the odd orders, 5 x 5 - [illegible] x 7 - 9 x 9 this association of complementary pairs [illegible] necessary to [illegible] the [illegible] forms. In the 8 x 8 size however, it bars the knight and Frankl[illegible] qualiti[illegible] therefore [illegible]s no particular interest.

[illegible] can be transformed [illegible]

KNIGHT

1	2	3	4	8	7	6	5
8	7	6	5	1	2	3	4

(800)

FRANKLIN-KNIGHT

1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3

(801)

FRANKLIN- JAINA

1	2	8	7	4	3	5	6
8	7	1	2	5	6	4	3

(802)

KNIGHT

1	4	6	7	8	5	3	2
8	5	3	2	1	4	6	7

(803)

ASSOCIATED

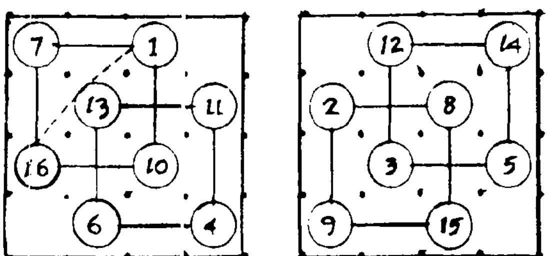
1	4	6	[illegible]	7	6	4	1
8	5	3	[illegible]	2	3	5	8

(804)

G				H			
1	2	3	4	1	2	4	3
1	2	5	6	1	2	6	5
1	3	5	7	1	3	7	5

(805)

THE CONSTRUCTION OF THE ORIGINAL JAINA SQUARE



(806)

KNIGHT TEST

1 1

1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3

2 2

1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3

FRANKLIN TEST (807)

A

1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	4
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3

B

1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3
1	2	4	3	8	7	5	6
8	7	5	6	1	2	4	3

COMBINATION OF (802) AND (803)

11	84	16	87	18	85	13	82
28	75	23	72	21	74	26	77
81	14	86	17	88	15	83	12
78	25	73	22	71	24	76	27
41	54	46	57	48	55	43	52
38	65	33	62	31	64	36	67
51	44	56	47	58	45	53	42
68	35	63	32	61	34	66	37

(808)

COMBINATION OF (801) AND (802)

11	82	18	87	14	83	15	86
28	77	21	72	25	76	24	73
41	52	48	57	44	53	45	56
38	67	31	62	35	66	34	63
81	12	88	17	84	13	85	16
78	27	71	22	75	26	74	23
51	42	58	47	54	43	55	46
68	37	61	32	65	36	64	33

(809)

THE FRANKLIN-KNIGHT-NASIK

EULER-LAHIREIAN MAGIC SQUARES

- - - - - - - - - - -

The magic squares numbers (810) (812) (813) are so constructed that any eight numbers in one direction as in the moves of a bishop or a knight are alike. The move of the bishop, starting anywhere, is what the Rev. A. H. Frost termed NASIK. Franklin broke this movement in the middle, and termed it a "bent-diagonal" and writers frequently simply call a square "Franklin," meaning it has this quality.

The move of the knight is shown in (814). The complementary pairs are in diagonally opposite subsquares and by turning subsquares two and four one-half around the position of the pairs is changed to what - in the odd orders - we term associated. "Diametrically equidistant from the center." See (815).

All of these squares have the maximum number of "centers of equilibrium" - 49 -. In all of them there are 207 squares and rectangles of even dimensions the four corner numbers of which bear a sum one-half that of rows or columns.

In (810) the rectangle No. 11 was used to form the primary B and then turned left (TL) to form the primary A. The same with (811) using the Franklin rectangle 011-015. The square (812) used the rectangle No. 15 for the B primary and No. 16 for the A primary, the latter being turned RIGHT. Two rectangles were also used for (813) one of them - No. 51 - being turned left for the A primary.

These four are all "Franklin" and contain 20 of the bent-diagonals "A" and 24 each of the "B" and "C". Many will contain the knight pattern "G" - for example - (813) even if they are not "knight."

Especially to be noted is the formation of the rectangles Nos. 111 to 165. The four digits - 1467 - and their complements - 8532 - comprising the upper and lower half of each 2 x 4 rectangle, bear the magic sum of 18 and therefore the rows and columns of each 4 x 4 subsquare in a magic square will be "magic," but not the diagonals.

Nº 11 (TL) AND Nº 11

71	25	76	22	78	24	73	27
38	64	33	67	31	65	36	62
41	55	46	52	48	54	43	57
88	14	83	17	81	15	86	12
21	75	26	72	28	74	23	77
68	34	63	37	61	35	66	32
51	45	56	42	58	44	53	47
18	84	13	87	11	85	16	82

(810)

Nº 011-015 (TL) AND Nº 011-016

71	25	78	24	76	22	73	27
38	64	31	65	33	67	36	62
21	75	28	74	26	72	23	77
68	34	61	35	63	37	66	32
41	55	48	54	46	52	43	57
88	14	81	15	83	17	86	12
51	45	58	44	56	42	53	47
18	84	11	85	13	87	16	82

(811)

Nº 16 (TR) AND Nº 15

11	85	17	83	18	84	12	86	
58	44	52	46	51	45	57	43	C
71	25	77	23	78	24	72	26	
38	64	32	66	31	65	37	63	
81	15	87	13	88	14	82	16	
48	54	42	56	41	55	47	53	A
21	75	27	73	28	74	22	76	
68	34	62	36	61	35	67	33	B

(812)

Nº 51 (TL) AND Nº 28

KNIGHT PATTERN &

67	33	61	35	62	36	68	34
52	46	58	44	57	43	51	45
77	23	71	25	72	26	78	24
82	16	88	14	87	13	81	15
37	63	31	65	32	66	38	64
42	56	48	54	47	53	41	55
27	73	21	75	22	76	28	74
12	86	18	84	17	83	11	85

(813)

Nº 151 (TL) AND Nº 111

31	64	36	67	38	65	33	62
58	45	53	42	51	44	56	47
21	74	26	77	28	75	23	72
88	15	83	12	81	14	86	17
61	34	66	37	68	35	63	32
48	55	43	52	41	54	46	57
71	24	76	27	78	25	73	22
18	85	13	82	11	84	16	87

(814)

TURN 2ND AND 4TH SUBSQUARES ONE-HALF AROUND TO MAKE THIS ASSOCIATED SQUARE

Nº 91 (TL) AND Nº 71

31	64	36	67	17	86	14	81
58	45	53	42	72	23	75	28
21	74	26	77	47	56	44	51
88	15	83	12	62	33	65	38
61	34	66	37	87	16	84	11
48	55	43	52	22	73	25	78
71	24	76	27	57	46	54	41
18	85	13	82	32	63	35	68

(815)

THE 2 x 2 UNIT - 1467 - METHOD OF CONSTRUCTING MAGIC SQUARES WITH THE FRANKLIN BENT DIAGONALS

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This method has many advantages over the one using magic rectangles. It has been possible to explore six different methods of placing the complementary pairs 11-88 12-87 13-86, etc., including the one used by Franklin which we have termed "original" and which cannot be accomplished with a single magic rectangle.

A great variety is possible as the start can be made with any one of the 48 units in (816) as long as this order is followed: inversion, reflection, reflection inverted entered zig-zag as indicated with the complements opposite.

Each 4 x 8 column forms the left half of a Franklin-knight-Nasik primary square the two 2 x 8 columns being transposed to form the right half (817). It has been reflected and turned one-quarter to the left, (turned on its down-right diagonal) to form the primary B square (818).

The Franklin-Jaina form is made by transposing the second and third 2 x 4 sections of (1) (7) to form the left half of the primary A and then transposing the two 2 x 8 columns to form the right half. These two primary squares (819) and (820) have the same construction as the thirteenth century JAINA in (806). The complementary pairs 1-8 2-7 3-6 4-5 are in the diagonally opposite corners of each 3 x 3 square within each 4 x 4 subsquare.

	(1)		(2)		(3)		(4)		(5)		(6)	
(1)	1 4	5 8	1 4	5 8	1 o	3 8	[illegible] 6	3 8	1 7	2 8	1 7	2 8
	6 7	2 3	7 6	3 2	4 7	2 5	7 4	5 2	4 6	3 5	6 4	5 3
(2)	2 3	6 7	3 2	7 6	2 5	4 7	5 2	7 4	3 5	4 6	5 3	6 4
	5 8	1 4	5 8	1 4	3 8	1 6	3 8	1 6	2 8	1 7	2 8	1 7

	(7)		(8)		(9)		(10)		(11)		(12)	
(3)	4 1	8 5	4 1	8 5	6 1	8 3	6 1	8 3	7 1	8 2	7 1	8 2
	7 6	3 2	6 7	2 3	7 4	5 2	4 7	2 5	6 4	5 3	4 6	3 5
(4)	3 2	7 6	2 3	6 7	5 2	7 4	2 5	4 7	5 3	6 4	3 5	4 6
	8 5	4 1	8 5	4 1	8 3	6 1	8 3	6 1	8 2	7 1	8 2	7 1

(816)

FRANKLIN-KNIGHT-NASIK

1	4	5	8	5	8	1	4
6	7	2	3	2	3	6	7
2	3	6	7	6	7	2	3
5	8	1	4	1	4	5	8
4	1	8	5	8	5	4	1
7	6	3	2	3	2	7	6
3	2	7	6	7	6	3	2
8	5	4	1	4	1	8	5

(817)

1	6	2	5	4	7	3	8
4	7	3	8	1	6	2	5
5	2	6	1	8	3	7	4
8	3	7	4	5	2	6	1
5	2	6	1	8	3	7	4
8	3	7	4	5	2	6	1
1	6	2	5	4	7	3	8
4	7	3	8	1	6	2	5

(818)

"ORIGINAL" FRANKLIN-NASIK

SAME AS (817)

(823)

1 4 5 8 6 7 2 3 4 1 8 5 7 6 3 2	REPEAT
REPEAT	REPEAT

(824)

JAINA-FRANKLIN-NASIK

1	4	5	8		8	1	4
6	7	2	3	2	3	6	7
4	1	8	5	8	5	4	1
7	6	3	2	3	2	7	6
2	3	6	7	6	7	2	3
5	8	1	4	1	4	5	8
3	2	7	6	7	6	3	2
8	5	4	1	4	1	8	5

(819)

1	6	4	7	2	5	3	8
4	7	1	6	3	8	2	5
5	2	8	3	6	1	7	4
8	3	5	2	7	4	6	1
5	2	8	3	6	1	7	4
8	3	5	2	7	4	6	1
1	6	4	7	2	5	3	8
4	7	1	6	3	8	2	5

(820)

"OPPOSITE" FRANKLIN-NASIK

1 4 5 8 6 7 2 3 2 3 6 7 5 8 1 4	REPEAT
4 1 8 5 7 6 3 2 3 2 7 6 8 5 4 1	REPEAT

(825)

SAME AS (818)

1	6	2	5	4	7	3	8
4	7	3	8	1	6	2	5
5	2	6	1	8	3	7	4
8	3	7	4	5	2	6	1
5	2	6	1	8	3	7	4
8	3	7	4	5	2	6	1
1	6	2	5	4	7	3	8
4	7	3	8	1	6	2	5

(826)

"DIRECT" FRANKLIN-NASIK

1	4	5	8	5	8	1	4
5	8	1	4	1	4	5	8
2	3	6	7	6	7	2	3
6	7	2	3	2	3	6	7
3	2	7	6	7	6	3	2
7	6	3	2	3	2	7	6
4	1	8	5	8	5	4	1
8	5	4	1	4	1	8	5

(821)

1	5	2	6	3	7	4	8
4	8	3	7	2	6	1	5
5	1	6	2	7	3	8	4
8	4	7	3	6	2	5	1
5	1	6	2	7	3	8	4
8	4	7	3	6	2	5	1
1	5	2	6	3	7	4	8
4	8	3	7	2	6	1	5

(822)

"ADJACENT" FRANKLIN-NASIK

4	5	8	1	8	1	4	5
7	2	3	6	3	6	7	2
3	6	7	2	7	2	3	6
8	1	4	5	4	5	8	1
1	8	5	4	5	4	1	8
6	3	2	7	2	7	6	3
2	7	6	3	6	3	2	7
5	4	1	8	1	8	5	4

(827)

4 5 8 1 7 2 3 6 1 8 5 4 6 3 2 7	REPEAT
REPEAT	REPEAT

(828)

To form the primary A for the "Direct" Franklin (821) the rows of (817) are rearranged in the order 1-4-3-2-7-6-5-8 which brings the complementary pairs together in each 2 x 2 section. This one also was turned on its down-right diagonal to form the primary B square (822).

The position of the complementary pairs in the 8 x 8 square which Franklin produced to illustrate his bent diagonals, is described as "adjacent and constructively adjacent" in each vertical half of the square. We have used the primary A of the Franklin-knight-Nasik (817) and one of the 4 x 4 subsquares of (819) - used four times for the primary B (823) (824).

In the Euler-Lahireian magic squares (829) and (833) the left halves are identical. The second and fourth subsquares of (829) are transposed and then reflected forming the square (833). It was necessary to form this square with "opposite" pairing in order to duplicate the style of pairing used by Franklin in his 16 x 16 square. In our example (856-A) as will be seen by examining the 8 x 8 Euler-Lahireian magic square (856-C) a different set of primary squares than (825) and (826) were necessary in order to bring the numbers 11-88 at the center bottom of the square.

In the "Adjacent" form, the primary A (817) is changed by placing the first column after the fourth and the fifth column after the eighth to form the primary A (827). The same proceeding with (824) to produce (828).

In forming these examples we have used only the first column (1) (7) in order to make certain comparisons. In (832-C) and (832-E) we have two primary squares quite different from the (823) and (824) both sets being used to produce the "original" Franklin.

FRANKLIN-KNIGHT

11	46	52	85	54	87	13	48
64	77	23	38	21	36	62	75
25	32	66	71	68	73	27	34
58	83	17	44	15	42	56	81
45	12	86	51	88	53	47	14
78	63	37	24	35	22	76	61
31	26	72	65	74	67	33	28
84	57	43	18	41	16	82	55

(829)

"ORIGINAL" FRANKLIN

11	44	55	88	51	84	15	48
66	77	22	33	26	37	62	73
24	31	68	75	64	71	28	35
57	86	13	42	17	46	53	82
41	14	85	58	81	54	45	18
76	67	32	23	36	27	72	63
34	21	78	65	74	61	38	25
87	56	43	12	47	16	83	52

(832)

JAINA-FRANKLIN

11	46	54	87	52	85	13	48
64	77	21	36	23	38	62	75
45	12	88	53	86	51	47	14
78	63	35	22	37	24	76	61
25	32	68	73	66	71	27	34
58	83	15	42	17	44	56	81
31	26	74	67	72	65	33	28
84	57	41	16	43	18	82	55

(830)

"OPPOSITE" FRANKLIN

11	46	52	85	14	47	53	88
64	77	23	38	61	76	22	35
25	32	66	71	28	33	67	74
58	83	17	44	55	82	16	41
45	12	86	51	48	13	87	54
78	63	37	24	75	62	36	21
31	26	72	65	34	27	73	68
84	57	43	18	81	56	42	15

(833)

"DIRECT" FRANKLIN

11	45	52	86	53	87	14	48
54	88	13	47	12	46	51	85
25	31	66	72	67	73	28	34
68	74	27	33	26	32	65	71
35	21	76	62	77	63	38	24
78	64	37	23	36	22	75	61
41	15	82	56	83	57	44	18
84	58	43	17	42	16	81	55

(831)

ADJACENT FRANKLIN

44	55	88	11	84	15	48	51
77	22	33	66	37	62	73	26
31	68	75	24	71	28	35	64
86	13	42	57	46	53	82	17
14	85	58	41	54	45	18	81
67	32	23	76	27	72	63	36
21	78	65	34	61	38	25	74
56	43	12	87	16	83	52	47

(834)

POSITION EQUIVALENTS

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The series of Lahireian numbers used by Leonhard Euler naturally runs from 11 to 99 and in squares of the eighth order it is only necessary to eliminate the ninth row and the ninth column, leaving the numbers 11 to 88 shown in figure (860). The position of each number is carried in the number itself - the first digit indicating the ROW and the second digit indicating the COLUMN.

Therefore any Euler-Lahireian magic square can be used as a KEY square in order to locate the proper substitute, in any series of numbers - not necessarily consecutive - for the Euler-Lahireian number.

For example: the second number in the KEY square (832-A) is 76 or row 7 and column 6 and the number in this position in the series of consecutive numbers (832-G) is 54 which is then entered in the square (832-B) which at the start is an EMPTY cell block. The next Euler-Lahireian number is 23 or row 2 and column 3 and its position equivalent in the series above is 11. This substitution is continued until the cell block is full and the magic square completed.

The original method of de La Hire is illustrated in (832-C) (832-D) (832-E) and can be used any time the student desires to have the magic square appear in consecutive numbers 1 to 64.

There is continuity - of a sort - in the entry of the first 32 numbers in this magic square with the original Franklin pairings (832-B) but doubtless some enthusiastic experimenter will find a better formula and - let us hope - the rule or rules governing the method of direct entry, which, unfortunately, was not disclosed by the many sided Benjamin Franklin.

								ROW
1	2	3	4	5	6	7	8	1
9	10	11	12	13	14	15	16	2
17	18	19	20	21	22	23	24	3
25	26	27	28	29	30	31	32	4
33	34	35	36	37	38	39	40	5
41	42	43	44	45	46	47	48	6
49	50	51	52	53	54	55	56	7
57	58	59	60	61	62	63	64	8
1	2	3	4	5	6	7	8	←(COLUMN)

(832-G)

KEY SQUARE

11	76	23	88	13	78	21	86
83	28	71	16	81	26	73	18
72	15	84	27	74	17	82	25
24	87	12	75	22	85	14	77
35	52	47	64	37	54	45	62
67	44	55	32	65	42	57	34
56	31	68	43	58	33	66	41
48	63	36	51	46	61	38	53

(832-A)

1	54	11	64	3	56	9	62
59	16	49	6	57	14	51	8
50	5	60	15	52	7	58	13
12	63	2	53	10	61	4	55
21	34	31	44	23	36	29	42
47	28	37	18	45	26	39	20
38	17	48	27	40	19	46	25
32	43	22	33	30	41	24	35

(832-B)

PRIMARY A and ITS ROOT VALUES + PRIMARY B

1	7	2	8	SAME AS LEFT
8	2	7	1	
7	1	8	2	
2	8	1	7	
3	5	4	6	SAME AS LEFT
6	4	5	3	
5	3	6	4	
4	6	3	5	

(832-C)

0	48	8	56
56	8	48	0
48	0	56	8
8	56	0	48
16	32	24	40
40	24	32	16
32	16	40	24
24	40	16	32

(832-D)

1	6	3	8	3	8	1	6
3	8	1	6	1	6	3	8
2	5	4	7	4	7	2	5
4	7	2	5	2	5	4	7
5	2	7	4	7	4	5	2
7	4	5	2	5	2	7	4
6	1	8	3	8	3	6	1
8	3	6	1	6	1	8	3

(832-E)